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On necessary conditions for infinite-dimensional extremum problems

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Abstract. In this paper, we carry on the analysis (introduced in [4] and developed in [2, 7]) of optimality conditions for extremum problems having infinite-dimensional image, in the case of unilateral constraints. This is done by associating to the feasible set a special multifunction. It turns out that the classic Lagrangian multiplier functions can be factorized into a constant term and a variable one; the former is the gradient of a separating hyperplane as introduced in [4, 5]; the latter plays the role of selector of the above multifunction. Finally, the need of enlarging the class of Lagrangian multiplier functions is discussed.

Key words: Image space, Lagrange multipliers, Multifunctions, Necessary optimality conditions, Nonsmooth optimization

1. Introduction

The use of separation arguments in the definition of optimality conditions for a constrained extremum problem has stressed the importance of the analysis of the image space associated to the given problem, defined as the product space where the images of the objective and the constraining functions run. The optimality of a feasible point is expressed by means of the disjunction of suitable subsets of the image space and it is proved using separation techniques. One main point in the development of the analysis is represented by the dimension of the image space, which can be finite or infinite, according to the nature of the constraints. The finite-dimensional case has been widely studied and many of the results obtained in this context can be generalized to an infinite dimensional problem under suitable additional assumptions (for example, that one of the sets that we need to separate, has non empty interior); in Section 2 we will recall the main features of the classic image space approach.

In the following part of the paper, we adopt a different point of view and we analyse the possibility of associating a finite-dimensional image space to a general infinite-dimensional extremum problem. This is done considering the constraints having an infinite-dimensional image, as multifunctions with values given by suitable subsets of a finite-dimensional space.

The existence of a selection, for the image multifunction, whose range has an empty intersection with a suitable subset of the image space, is a necessary and sufficient optimality condition for the given extremum problem. Under the hypothesis of continuity of the constraint and objective functions, the selection is defined as a weighted integral where the weights are closely related to the classic Lagrange multipliers. For this reason, these weights will be referred to as "selection multipliers" and will be considered as an enlargement of the class of multipliers associated to the problem. In the hypothesis where the selection multipliers do not locally depend on the variable x of the problem, the classic results of Calculus of Variations can be recovered.

We recall the main notations that will be used in the sequel.

Let the positive integers n, m, p, with $p \leqslant m$, the interval $T := [a, b] \subset \mathrm{IR}$ with $-\infty < a < b < +\infty$ and the functions $\psi_i : \mathrm{IR} \times \mathrm{IR}^n \times \mathrm{IR}^n \to \mathrm{IR}, i = 0, \dots, m$, be given. Let Y be the set of all continuous functions $x(t) := (x_1(t), \dots, x_n(t)), t \in T$, with continuous derivatives $x'(t) := (x_1'(t), \dots, x_n'(t))$, $t \in T$, except at most a finite number of points $\bar{t}$ at which there exist and are finite $\lim_{t \downarrow \bar{t}} x'(t)$ and $\lim_{t \uparrow \bar{t}} x'(t)$. Let us define $x'(\bar{t}) = \lim_{t \downarrow \bar{t}} x'(t)$. The set Y forms a vector space on the set of real numbers. The space Y will be equipped with the norm

$$||x||_\infty := \max_{t \in T} ||x(t)||, \quad x \in Y,$$

where $|| \bullet ||$ denotes the Euclidean norm of IR^n. Set $\ell := \{1, \dots, m\}$.

Taking for granted the results of [2], we continue the study of problem (2) of [2] in the case of unilateral constraints only, and we consider a particular problem belonging to the following class:

$$\min f(x), \text{ s.t. } g_i(x) \geqslant 0, \ i \in \ell, \ x \in X, \tag{1.1}$$

namely, the problem:

$$\min f(x) := \int_T \psi_0(t, x(t), \ x'(t)) \mathrm{d}t, \tag{1.2a}$$

subject to:

$$\psi_i(t, x(t), \ x'(t)) \geqslant 0, \ \forall t \in T, \ i \in \ell, \tag{1.2b}$$

$$x \in X := \{x \in Y : x_i(a) = x_i(b) = 0, \ i \in \ell\}. \tag{1.2c}$$

2. The Image Space Approach

In this section we present the main features of the image space analysis following the approach developed in [3]. The main idea of the image space analysis is to transform the given constrained extremum problem, into separation problems in the image space, defined by the space where the objective and the constraint functions run. The optimality of a feasible point is equivalent to the disjunction of suitable subsets of the image space. We will see that the image space associated to (1.2) is infinite-dimensional, which is the main novelty in the analysis of the image space approach developed in this section.

Consider the function $F: Y \longrightarrow \mathbb{R} \times C(T)^m$, defined by
$F(x) := (f(\overline{x}) - f(x), g_i(x), \ \ i \in \ell)$, where $\overline{x} \in X$ is a feasible point and $g_i := \psi_i$, where $\psi_i : Y \longrightarrow C(T)$ is now considered as a function of x, for $i \in \ell$.

Define

$$\mathcal{K}(\overline{x}) := \{(u, v) \in \mathbb{R} \times C(T)^m : u = f(\overline{x}) - f(x), \ v_i = g_i(x), \ i \in \ell, \ x \in X\},$$

and

$$\mathcal{H} := \{(u, v) \in \mathbb{R} \times C(T)^m : u > 0, v_i \geqslant 0, \ i \in \ell\},$$

where $v_i \geqslant 0$ means that $v_i(t) \geqslant 0, \forall t \in T, i \in \ell$.

The set $\mathcal{K}(\overline{x})$ and the space $\mathbb{R} \times C(T)^m$ are called the *image* and the *image space* associated to the problem (1.2), respectively. In the present case, the extremum problem (1.2) is defined in the infinite-dimensional space Y and possesses an infinite-dimensional image. Anyway, there are several problems, as for example those of isoperimetric type, that are defined in an infinite-dimensional space but have a finite-dimensional image: for the latter problems, the analysis in the image space, developed in the finite-dimensional case, is still valid.

The optimality of $\overline{x}$ is expressed by the disjunction of the sets $\mathcal{K}(\overline{x})$ and $\mathcal{H}$.

LEMMA 2.1. *$\bar{x}$ is an optimal solution for* (1.2) *iff*

$$\mathcal{K}(\overline{x}) \cap \mathcal{H} = \varnothing \tag{2.1}$$

Proof. It is a consequence of the fact that $\bar{x}$ is an optimal solution for (1.2) iff the following system is impossible

$$f(\overline{x}) - f(x) > 0, g_i(x) \geqslant 0, \ x \in X, i \in \ell,$$

which is equivalent to (2.1). □

In order to obtain a local version of the previous lemma, it is enough to replace, in the definition of the set $\mathcal{K}(\overline{x})$, the condition "$x \in X$" with "$x \in X \cap N(\overline{x})$", where $N(\overline{x})$ is a neighbourhood of $\overline{x}$.

To prove directly whether $\mathcal{K}(\overline{x}) \cap \mathcal{H} = \emptyset$ or not, is, in general, very difficult; therefore, such a disjunction will be proved showing that the two sets, or the set $\mathcal{H}$

and an extension of the image depending on $\mathscr{H}$, lie in two disjoint level sets of a suitable functional; when the functional can be found linear $\mathscr{K}(\bar{x})$ and $\mathscr{H}$ will be said "linearly separable".

The image $\mathscr{K}$ may not be convex , in general, even when the given extremum problem is convex (i.e., the functions f and $-g$ are convex). In order to overcome this difficulty, a regularization of the image set, called the *conic extension with respect to the cone $cl\mathscr{H}$* and denoted by $\mathcal{E}(\bar{x})$, has been introduced in the following form:

$$\mathcal{E}(\bar{x}) = \mathscr{K}(\bar{x}) - cl\mathscr{H}, \tag{2.2}$$

or, equivalently

$$\mathcal{E}(\bar{x}) := \{(u,v) \in \mathrm{IR} \times C(T)^m : u \leqslant f(\bar{x}) - f(x),\ v_i \leqslant g_i(x), i \in \ell,\ x \in X, \},$$

The importance of the conic extension of the image is ensured by the following statement:

PROPOSITION 2.1. $\mathscr{K}(\bar{x}) \cap \mathscr{H} = \emptyset$ *iff*

$$\mathcal{E}(\bar{x}) \cap \mathscr{H} = \emptyset. \tag{2.3}$$

Proof. It is simple to prove that it results $\mathscr{H} + cl\mathscr{H} = \mathscr{H}$. Therefore, (2.3) is a direct consequence of the relations

$$\mathcal{E}(\bar{x}) - \mathscr{H} = \mathscr{K}(\bar{x}) - cl\mathscr{H} - \mathscr{H} = \mathscr{K}(\bar{x}) - (\mathscr{H} + cl\mathscr{H}) = \mathscr{K}(\bar{x}) - \mathscr{H}. \qquad \square$$

Hence, (2.3) is a necessary and sufficient optimality condition for the problem (1.2).

In certain cases, it is easier to prove (2.3) because the conic extension may have some advantageous properties that $\mathscr{K}(\bar{x})$ has not. In the case of convex optimization, the conic extension is a convex set so that the Hahn–Banach separation theorem can be used.

PROPOSITION 2.2. *Let $\psi_0(t, x(t), x'(t))$ and $-\psi_i(t, x(t), x'(t))$ be convex with respect to the second and the third argument jointly,* $\forall i \in \ell$. *Then $\mathcal{E}(\bar{x})$ is a convex set.*

Proof. Let $(u^1, v^1), (u^2, v^2) \in \mathcal{E}(\bar{x})$; therefore, there exist $x^1, x^2 \in X$ such that

$$u^1 \leqslant f(\bar{x}) - f(x^1), v^1 \leqslant g(x^1), u^2 \leqslant f(\bar{x}) - f(x^2), v^2 \leqslant g(x^2),$$

where $f(x) = \int_T \psi_0(t, x(t), x'(t))\,\mathrm{d}t$ and $g(x) = (\psi_i(t, x(t), x'(t)), i \in \ell)$.

We have to prove that, $\forall \alpha \in [0, 1]$

$$\alpha(u^1, v^1) + (1-\alpha)(u^2, v^2) \in \mathcal{E}(\bar{x}). \tag{2.4}$$

We have

$$\alpha u^1+(1-\alpha)u^2\leqslant \alpha f(x^1)+(1-\alpha)f(x^2)=$$
$$\alpha\int_T[\psi_0(t,\bar{x},\bar{x}')-\psi_0(t,x^1,(x^1)')]\mathrm{d}t+$$
$$(1-\alpha)\int_T[\psi_0(t,\bar{x},\bar{x}')-\psi_0(t,x^2,(x^2)')]\mathrm{d}t=$$
$$\int_T[\psi_0(t,\bar{x},\bar{x}')-\alpha\psi_0(t,x^1,(x^1)')+(1-\alpha)\psi_0(t,x^2,(x^2)')]\mathrm{d}t\leqslant$$
$$\int_T[\psi_0(t,\bar{x},\bar{x}')-\psi_0(t,\alpha x^1+(1-\alpha)x^2,\alpha(x^1)'+(1-\alpha)(x^2)')]\mathrm{d}t,$$

where the last inequality is due to the convexity of $\psi_0(t,\cdot,\cdot)$.

Similarly, $\forall i\in\ell$, we have

$$\alpha v_i^1+(1-\alpha)v_i^2\leqslant \alpha g_i(x^1)+(1-\alpha)g_i(x^2)\leqslant g_i(\alpha x^1+(1-\alpha)x^2),$$

taking into account that the hypotheses guarantee the convexity of the function $-g$. Since X is a convex set then $\alpha x^1+(1-\alpha)x^2\in X$ so that (2.4) holds, which completes the proof of the Proposition. □

It is easy to show that the cone $\mathcal{H}$ is convex and with a nonempty interior. This allows us to prove that the sets $\mathcal{E}(\bar{x})$ and $\mathcal{H}$, if disjoint, are linearly separable.

THEOREM 2.1 (Linear separation). *Let $\bar{x}$ be an optimal solution for* (1.2) *and let $\psi_0(t,x(t),x'(t))$ and $-\psi_i(t,x(t),x'(t))$ be convex with respect to the second and the third argument jointly, $\forall i\in\ell$. Then there exist a number $\theta\geqslant 0$ and functions μ_i, $i\in\ell$, of bounded variation, such that $(\theta,\mu_i,i\in\ell)\neq 0$ and*

$$\theta u+\sum_{i\in\ell}\int_T v_i(t)d\mu_i(t)\leqslant 0,\quad \forall(u,v)\in\mathcal{E}(\bar{x}).\tag{2.5}$$

Proof. The proof is a consequence of the Hahn-Banach separation theorem, taking into account that, in the considered hypotheses, $\mathcal{E}(\bar{x})$ and $\mathcal{H}$ are disjoint convex sets, the cone $\mathcal{H}$ has a nonempty interior. Recalling that every continuous linear functional h on the space $C(T)^m$ can be uniquely represented in the form

$$\langle h,y(\cdot)\rangle=\sum_{i=1}^m\int_T y_i(t)d\mu_i(t),$$

where μ_i are functions of bounded variation, for $i=1,\dots,m$. Then a continuous linear functional on $\mathrm{IR}\times C(T)^m$ takes the form

$$\theta u+\sum_{i=1}^m\int_T y_i(t)d\mu_i(t)=\beta,$$

where $\theta, \beta \in \mathrm{IR}$. Without loss of generality, we can assume that

$$\theta u + \sum_{i=1}^{m} \int_T v_i(t) d\mu_i(t) \geqslant \beta, \quad \forall (u,v) \in \mathcal{H}. \tag{2.6}$$

We observe that it must be $\beta \leqslant 0$, since $0 \in cl\mathcal{H}$, and it must be $\theta \geqslant 0$ since $\{(u,0), u>0\} \subset \mathcal{H}$. Moreover, we have

$$\theta u + \sum_{i=1}^{m} \int_T v_i(t) d\mu_i(t) \leqslant \beta, \quad \forall (u,v) \in \mathcal{E}(\overline{x}). \tag{2.7}$$

We have to prove that $\beta = 0$.

To this end, observe that

$$\theta u + \sum_{i=1}^{m} \int_T v_i(t) d\mu_i(t) \geqslant 0, \quad \forall (u,v) \in \mathcal{H}. \tag{2.8}$$

Actually, if there exists $(u^*, v^*) \in \mathcal{H}$ such that

$$\theta u^* + \sum_{i=1}^{m} \int_T v_i^*(t) d\mu_i(t) = -\gamma \ (\gamma > 0),$$

then, since $\lambda(u^*, v^*) \in \mathcal{H}, \forall \lambda > 0$, it would be

$$\lim_{\lambda \to \infty} \theta \lambda u^* + \lambda \sum_{i=1}^{m} \int_T v_i^*(t)\ d\mu_i(t) = -\infty,$$

which is against (2.6). It is immediate that (2.8) also holds $\forall (u,v) \in cl\mathcal{H}$. Choose $\bar{u} = f(\overline{x}) - f(\overline{x}) = 0$ and $\bar{v} = g(x) \geqslant 0$ so that $(\bar{u}, \bar{v}) \in \mathcal{E}(\overline{x}) \cap cl\mathcal{H}$. By (2.7), we have

$$\beta \geqslant \theta \bar{u} + \sum_{i=1}^{m} \int_T \bar{v}_i(t)\ d\mu_i(t) \geqslant 0,$$

so that $\beta = 0$. □

In the original space X, the condition (2.5) becomes

$$\theta(f(\overline{x}) - f(x)) + \sum_{i \in \ell} \int_T \psi_i(t, x(t), x'(t)) d\mu_i(t) \leqslant 0, \quad \forall x \in X. \tag{2.9}$$

It can be shown that (2.9) is equivalent to the classic saddle point condition for the Lagrangian function associated to the given problem (1.2).

The linear separation in the image space has been widely studied in [8] where the image space approach has been employed in the analysis of generalized systems.

3. The Multifunction Approach: Selection Quasi-multipliers

In the previous section, we have introduced the image associated to (1.2) by means of the function F, defined by (2.1). In the present section, we consider the same map as a multifunction $F: Y \rightrightarrows \mathbb{R}^{1+m}$, where
$F(x) := (f(\overline{x}) - f(x), \underset{t\in T}{\cup} \psi_i(t, x(t), x'(t)), i \in \ell))$. Similarly, we define

$$\mathscr{H} := \{(u, v) \in \mathbb{R} \times \mathbb{R}^m \ : \ u > 0, v \geqslant 0\}.$$

$F(x)$ is now a set, not necessarily a singleton. Thus the optimality cannot be expressed by the disjunction of $\mathscr{H}$ and $F(X)$. It is known [2] that $\bar{x}$ is a local optimal solution for (1.2) iff

$$F(x) \not\subset \mathscr{H}, \ \forall x \in N(\bar{x}). \tag{3.1}$$

By selecting an element from $F(x)$ or from its convex hull, we may hope to reduce ourselves to the scheme of [5, 6], outlined in the previous section. The previous considerations lead to the following lemma.

LEMMA 3.1. *$\bar{x}$ is a local optimal solution for* (1.2) *iff there exists a function* α: $X \to \mathbb{R}^{1+m}$ *such that*:

$$\alpha(x) \in F(x) \text{ and } \alpha(x) \notin \mathscr{H}, \quad \forall x \in N(\overline{x}), \tag{3.2}$$

where $N(\overline{x})$ is a neighbourhood of $\overline{x} \in Y$.

Proof. It is enough to observe that (3.1) is equivalent to (3.2). □

If we define

$$K_\alpha(\overline{x}) := \{(u, v) \in F(X) : (u, v) = \alpha(x), x \in N(\overline{x})\},$$

then (3.2) is equivalent to

$$K_\alpha(\overline{x}) \cap \mathscr{H} = \varnothing \tag{3.3}$$

$\mathscr{K}_\alpha(\overline{x})$ will be called the *selected image*, and will play the same role as $\mathscr{K}$ in [5, 6].

The infinite dimensionality of the image is overcome by the selection: instead of considering the image of (1.2), which would lead us to an infinite-dimensional image space, we introduce the multifunction F, so that we have a finite-dimensional image space, where the scheme of [5, 6] can be adopted by replacing $\mathscr{K}$ with $K_\alpha(\overline{x})$.

Thus, by means of the selection, we can associate to the infinite-dimensional problem (1.2) an equivalent finite-dimensional problem.

LEMMA 3.2. *$\bar{x}$ is a local optimal solution for (1.2) iff there exists a selection function α for F such that $\bar{x}$ is a local minimum point of the problem*:

$$\min f(x), \ \textit{s.t.} \ \alpha_i(x) \geqslant 0, \quad i = 1, \ldots, m, \quad x \in X. \tag{3.4}$$

(3.4) will be called the selected problem associated to (1.2).

Proof. $\bar{x}$ is a local minimum point of (3.4) iff the system

$$f(\bar{x})-f(x)>0, \alpha_i(x)\geqslant 0, i=1,\ldots,m, \quad x\in N(\bar{x})$$

is impossible. Since, given $x\in X$, the first component u of the vector $(u,v)\in F(X)$ is uniquely defined, then necessarily it is

$$\alpha_0(x)=f(\bar{x})-f(x).$$

Then, the impossibility of the previous system is equivalent to (3.1) and, by Lemma 3.1, to the optimality of $\bar{x}$. □

The function α is a local selection of $F(x)$ in a neighbourhood of $\bar{x}$. A fundamental aspect of our analysis lies in the possibility of considering selection functions which belong to a suitable class.

Let us analyse the case where the selection is expressed by means of a weighted integration, namely:

$$\alpha_i(x)=\int_T \omega_i(t,x)\psi_i(t,x,x')\mathrm{d}t, \quad i\in\ell, \tag{3.5}$$

where $\omega_i: T\times X\to \mathrm{IR}$ $i\in\ell$, and $\omega:=(\omega_1,\ldots,\omega_m)\in\Omega$, Ω being a given set of parameters.

Consider the function $\Phi: X\times\Omega\to \mathrm{IR}^{1+m}$, defined, $\forall x\in X$, by:

$$\Phi(x,\omega):=\left(f(\bar{x})-f(x), \int_T \omega_i(x,t)\psi_i(t,x,x')\mathrm{d}t, \quad i\in\ell\right). \tag{3.6}$$

According to [2], Φ is called a *generalized selection function, iff*

$$F(x)\subseteq\mathcal{H}\Leftrightarrow\Phi(x,\omega)\in\mathcal{H}, \forall\omega\in\Omega;$$

ω *is a* selection quasi-multiplier (for short, SQM). Classically, in the literature, the multipliers ω depend only on the variable t; next example shows the need of enlarging the class of multipliers from $\omega_i(t)$ to $\omega_i(t;x)$.

EXAMPLE 3.1. Let us consider the problem:

$$\min\int_T \cos x(t)\mathrm{d}t, x(t)=0, \forall t\in T:=[0,1], \quad x\in Y. \tag{3.7}$$

Of course, $x(t)\equiv 0$ is the minimum point. The selected problem, defined by (3.4), becomes

$$\min\int_T \cos x(t)\mathrm{d}t, \int_T \omega(x,t)x(t)\mathrm{d}t=0, \quad x\in Y, \tag{3.8}$$

where $\omega(x,\cdot)\in L^1(T), \quad \forall x\in Y$.

We observe that, if we choose $\omega(x,t)=x(t)$, then the selected problem (3.8) is equivalent to (3.7).

On the contrary, we prove that $x(t)\equiv 0$ is not solution of (3.8) if ω is chosen independent on x, that is $\omega=\omega(t)$. Note that we have:

$$\int_T \cos x(t)\,\mathrm{d}t \leqslant 1.$$

We look for a solution of type

$$x(t)=at+b,\quad a\neq 0,\ b\neq 0;$$

Hence, we must have:

$$\int_T \omega(t)x(t)\,\mathrm{d}t = a\int_T \omega(t)t\,\mathrm{d}t + b\int_T \omega(t)\,\mathrm{d}t = 0.$$

We observe that $a,b\neq 0$ imply that $x(t)=at+b\equiv 0$. Now, let us evaluate the objective function in $x(t)$:

$$\int_T \cos(at+b)\,\mathrm{d}t = \frac{2}{a}\sin\frac{a}{2}\cos\frac{a+2b}{2}.$$

If the minimum were 1, then we should have:

$$\sin\frac{a}{2}\cos\frac{a+2b}{2}=\frac{a}{2}.$$

By choosing a and b in such a way that

$$a\int_T \omega(t)t\,\mathrm{d}t + b\int_T \omega(t)\,\mathrm{d}t = 0,\quad \frac{a}{2}\notin[-1,1],$$

the previous equality is false. □

As a consequence of the above example, we have that the approach of [6] cannot be extended, if the multiplier ω does not depend on x. Hence, we introduce a SQM depending on x; in this case we will obtain a necessary condition, like that of [6].

Let us analyse, more in details, the existence of selections which enjoy suitable properties. The continuity of the constraint and objective functions ensures the existence of a selection α which is continuous at $\overline{x}$.

PROPOSITION 3.1. *Let $\overline{x}$ be a minimum point of* (1.2), *and $\psi_i : C^1(T)\to C(T)$, $i=0,1,\dots,m$ be continuous in $N(\overline{x})$. Then, the function*

$$\alpha(x) := \{f(\overline{x})-f(x), \min_{t\in T}\psi_1(t,x(t),x'(t)),\dots,\min_{t\in T}\psi_m(t,x(t),x'(t))\}$$

is a selection function, which is continuous in $N(\overline{x})$.

Proof. We have to show that the functions:

$$f(\overline{x})-f(x)\quad\text{and}\quad \min_{t\in T}\psi_i(t,x(t),x'(t)),\ i\in\ell,$$

are continuous in $N(\overline{x})$. Let $\tilde{x} \in N(\overline{x})$, and $\varepsilon > 0$; we must prove the inequality:

$$\left| \int_T [\psi_0(t, \tilde{x}(t), \tilde{x}'(t)) - \psi_0(t, x(t), x'(t))] \, \mathrm{d}t \right| < \varepsilon, \quad \forall x \in U(\tilde{x}),$$

where $U(\tilde{x})$ is a neighbourhood of $\tilde{x}$. Since ψ_0 is continuous at $\tilde{x}, \exists \delta > 0$, such that the inequality

$$\sup_{t \in T} |\tilde{x}(t) - x(t)| < \delta \tag{3.9}$$

implies

$$\sup_{t \in T} |\psi_0(t, \tilde{x}(t), \tilde{x}(t)) - \psi_0(t, x(t), x'(t))| < \frac{\varepsilon}{b-a}.$$

Hence, for each x which fulfils (3.9), we have:

$$\left| \int_T [\psi_0(t, \tilde{x}(t), \tilde{x}'(t)) - \psi_0(t, x(t), x'(t))] \, \mathrm{d}t \right| < \int_T \frac{\varepsilon}{b-a} \mathrm{d}t = \varepsilon.$$

Let $i \in \ell$. Consider the inequality:

$$\left| \min_{t \in T} \psi_i(t, \tilde{x}(t), \tilde{x}'(t)) - \min_{t \in T} \psi_i(t, x(t), x'(t)) \right| < \varepsilon. \tag{3.10}$$

ψ_0 being continuous at $\tilde{x}, \exists \overline{\delta} > 0$, such that the inequality

$$\sup_{t \in T} |\tilde{x}(t) - x(t)| < \overline{\delta} \tag{3.11}$$

implies:

$$\sup_{t \in T} |\psi_i(t, \tilde{x}(t), \tilde{x}'(t)) - \psi_i(t, x(t), x'(t))| < \varepsilon.$$

Let

$$\psi_i(\overline{t}, \tilde{x}(\overline{t}), \tilde{x}'(\overline{t})) = \min_{t \in T} \psi_i(t, \tilde{x}(t), \tilde{x}'(t)),$$
$$\psi_i(t^0, x(t^0), x'(t^0)) = \min_{t \in T} \psi_i(t, x(t), x'(t)).$$

We have, $\forall t \in T$, :

$$\psi_i(\overline{t}, \tilde{x}(\overline{t}), \tilde{x}'(\overline{t})) - \psi_i(t, x(t), x'(t)) \leqslant \psi_i(t, \tilde{x}(t), \tilde{x}'(t) - \psi_i(t, x(t), x'(t)) < \varepsilon,$$

so that:

$$\psi_i(\overline{t}, \tilde{x}(\overline{t}), \tilde{x}'(\overline{t})) < \psi_i(t^0, x(t^0), x'(t^0)) + \varepsilon.$$

Similarly, $\forall t \in T$,

$$\psi_i(t^0, x(t^0), x'(t^0)) - \psi_i(t, \tilde{x}(t), \tilde{x}'(t)) \leqslant \psi_i(t, x(t), x'(t)) - \psi_i(t, \tilde{x}(t), \tilde{x}'(t)) < \varepsilon,$$

so that:

$$\psi_i(\bar{t},\tilde{x}(\bar{t}),\tilde{x}'(\bar{t})) > \psi_i(t^0,x(t^0),x'(t^0)) - \varepsilon.$$

Hence, for each x which fulfils (3.11), we have that (3.10) is satisfied. □

In the hypotheses of Proposition 3.1, it is possible to ensure the existence of a continuous generalized selection belonging to the class (3.6). Actually, if we put

$$\omega_i(t,x) := k_i(x)\psi_i(t,x,x'),$$

where

$$\begin{cases} 0, & \text{if} \quad \int_T [\psi_i(t,x,x')]^2 \mathrm{d}t = 0, \\ \dfrac{\min_{t\in T}\psi_i(t,x,x')}{\int_T [\psi_i(t,x,x')]^2 \mathrm{d}t}, & \text{otherwise} \end{cases}$$

for $i \in \ell$,

then it is easy to see that the generalized selection function Φ defined by (3.6) coincides with the selection function α defined in the Proposition 3.1.

A necessary optimality condition for (1.2) will be reached by extending the approach of [5, 6] to the selected problem (3.4), namely (1.2) where (1.2b) is replaced by

$$g_i(x;\omega_i) := \int_T \omega_i(t,x)\psi_i(t,x,x')\mathrm{d}t \geqslant 0, \quad i \in \ell. \tag{3.12}$$

As in [2, 5] the analysis will be carried out within the class of $\mathcal{C}$-differentiable functions; in the sequel the $\mathcal{C}$-derivative will be always assumed to be bounded (with respect to the 2-nd argument). Here there is a further difficulty: the $\mathcal{C}$-differentiability must be enjoyed by $f(x)$ and $g_i(x;\omega_i), i \in \ell$, and should be unsuitable to assume it; it is more appropriate that any assumption is made on the given data ψ_i and on the selection multiplier ω_i.

PROPOSITION 3.2. *Let* $f_i : X \to \mathrm{IR}, i = 1,2$ *be* $\mathcal{C}$*-differentiable at* $x = \bar{x}$ *and let*

$$f_i(x) = f_i(\bar{x}) + \mathscr{D}_{\mathcal{C}} f_i(\bar{x};z) + \epsilon_i(\bar{x};z), \quad i = 1,2$$

be their expansions, where $\mathscr{D}_{\mathcal{C}} f_i, i = 1,2$ *are the* $\mathcal{C}$*-derivatives. Set* $\tilde{f} := f_1 \cdot f_2$ *and assume that*:

$$\mathscr{D}_{\mathcal{C}} \tilde{f}(\bar{x};z) := \mathscr{D}_{\mathcal{C}} f_1(\bar{x};z) \cdot f_2(\bar{x}) + f_1(\bar{x}) \cdot \mathscr{D}_{\mathcal{C}} f_2(\bar{x};z) \in \mathcal{C}. \tag{3.13}$$

$\tilde{f}$ *is* $\mathcal{C}$*-differentiable at* $\bar{x}$ *in the direction* z *and its expansion is given by*

$$\tilde{f}(x) = \tilde{f}(x) + \mathscr{D}_{\mathcal{C}} \tilde{f}(\bar{x};z) + \tilde{\epsilon}(\bar{x};z), \tag{3.14}$$

where

$$\tilde{\epsilon}(\overline{x}; z) := \epsilon_1 \cdot \epsilon_2 + \epsilon_1 \cdot [f_2(\overline{x}) + \mathscr{D}_{\mathcal{C}} f_2] + \epsilon_2 [f_1(\overline{x}) + \mathscr{D}_{\mathcal{C}} f_1] + \mathscr{D}_{\mathcal{C}} f_1 \cdot \mathscr{D}_{\mathcal{C}} f_2. \quad (3.15)$$

Proof. The expansion of $\tilde{f}$ is trivially obtained from the product of the expansions of f_1 and f_2. Because of assumption (3.13) $\mathscr{D}_{\mathcal{C}} \tilde{f}$ is sublinear; hence we have to prove only that $\lim_{z \to 0} \tilde{\epsilon}/||z|| = 0$. As $z \to 0$, obviously $\epsilon_1 \cdot \epsilon_2/||z|| \to 0$; the same happens to the 2nd and 3rd terms in the RHS of (3.15), since the forms in square brackets are bounded. The boundedness of $\mathscr{D}_{\mathcal{C}} f_1/||z||$ and $\lim_{z \to 0} \mathscr{D}_{\mathcal{C}} f_2 = 0$ imply that $\mathscr{D}_{\mathcal{C}} f_1 \cdot \mathscr{D}_{\mathcal{C}} f_2/||z|| \to 0$ as $z \to 0$. This completes the proof. □

Assumption (3.13) is fulfilled, when f_1 and f_2 are differentiable, since $\mathscr{D}_{\mathcal{C}} f_i, i = 1, 2$ are linear (in this case $\mathscr{D}_{\mathcal{C}} \tilde{f} = \langle f_1'(\overline{x}) f_2(\overline{x}) + f_1(\overline{x}) f_2'(\overline{x}), z \rangle$, which is the classic formula), or when $\mathscr{D}_{\mathcal{C}} f_i, i = 1, 2$ are not linear and $f_i(\overline{x}) \geqslant 0, i = 1, 2$. When $f_i(\overline{x}) < 0$, then $\tilde{f}$ may not be $\mathcal{C}$-differentiable; see for instance the case where $f_1(x) = |x|, f_2(x) = |x| - 1, x \in \mathbb{R}$.

We will assume the $\mathcal{C}$-differentiability of $\psi_0, -\psi_1, i \in \ell$ with respect to the set of 2nd and 3rd arguments, of ω_i with respect to the 2nd argument and that all the hypotheses of Theorem 3.1 and 4.1 in [2] are satisfied. As a consequence we will have the following expansion (for the sake of simplicity, in the sequel $\overline{x}$ will be replaced merely by x):

$$f(x + \delta x) = f(x) + \int_T \mathscr{D}_{\mathcal{C}} \psi_0(t, x, x'; \delta x, \delta x') \mathrm{d}t + \int_T \epsilon_{\psi_0}(t, x, x'; \delta x, \delta x') \mathrm{d}t, \quad (3.16a)$$

$$g_i(x + \delta x; \omega_i) = g_i(x; \omega_i) + \int_T \mathscr{D}_{\mathcal{C}} \pi_i(t, x, x'; \delta x, \delta x') \mathrm{d}t + + \int_T \epsilon_i^{\pi}(t, x, x'; \delta x, \delta x') \mathrm{d}t, i \in \ell, \quad (3.16b)$$

where

$$\pi_i := \omega_i \cdot \psi_i; \mathscr{D}_{\mathcal{C}} \pi_i := \mathscr{D}_{\mathcal{C}} \omega_i(t, x; \delta x) \cdot \psi_i(t, x, x') + \omega_i(t, x) \cdot \mathscr{D}_{\mathcal{C}} \psi_i(t, x, x'; \delta x, \delta x');$$

$$\epsilon_i^{\pi} := \epsilon_{\omega_i} \cdot \epsilon_{\psi_i} + \epsilon_{\omega_i} \cdot [\psi_i(t, x, x') + \mathscr{D}_{\mathcal{C}} \psi_i] + \epsilon_{\psi_i} \cdot [\omega_i(t, x) + \mathscr{D}_{\mathcal{C}} \omega_i(t, x, \delta x)] + \mathscr{D}_{\mathcal{C}} \omega_i \cdot \mathscr{D}_{\mathcal{C}} \psi_i;$$

and where the pairs $(\mathscr{D}_{\mathcal{C}} \omega_i, \epsilon_{\omega_i}), (\mathscr{D}_{\mathcal{C}} \psi_i, \epsilon_{\psi_i})$ give the expansions of ω_i, ψ_i, respectively. Since $\mathscr{D}_{\mathcal{C}}$ is an operator which denotes $\mathcal{C}$-derivative, the use of $\mathscr{D}_{\mathcal{C}} \pi_i$ as a symbol would be improper; this does not happen here since π_i is $\mathcal{C}$-differentiable due to Proposition 3.2. When ω_i and ψ_i are differentiable $\mathscr{D}_{\mathcal{C}} \pi_i$ collapses to the usual derivative of a product. If ω_i is constant with respect to x, so that can be denoted by $\omega_i(t)$, then $\mathscr{D}_{\mathcal{C}} \pi_i = \omega_i(t) \cdot \mathscr{D}_{\mathcal{C}} \psi_i$ and $\epsilon_i^{\pi} = \epsilon_{\psi_i} \cdot [\omega_i(t, x) + \mathscr{D}_{\mathcal{C}} \omega_i(t, x, \delta x)]$.

4. Homogeneization

For the sake of simplicity, in this section we will assume that the selection multipliers (for short, SM) ω_i do not depend on x, but only on t, so that the selection function α is defined by

$$\alpha_i(x) = \int_T \omega_i(t)\psi_i(t,x,x')\mathrm{d}t, \quad i \in \ell \tag{4.1}$$

for $x \in N(\overline{x})$, a neighbourhood of $\overline{x}$.

Next proposition is a consequence of the above assumptions and of the results stated in Section 3.

PROPOSITION 4.1. *Assume that* $\alpha_i(x)$ *be defined by* (4.1), $i \in \ell$. *If the system*

$$f(\overline{x}) - f(x) > 0; \psi_i(t,x(t),x'(t)) \geqslant 0, i \in \ell, \forall t \in T, x \in X \cap N(\overline{x}), \tag{4.2a}$$

is impossible, iff, there exists $\omega \in \Omega$ *such that the following system is also impossible*:

$$f(\overline{x}) - f(x) > 0, g(x,\omega) \geqslant 0, x \in X \cap N(\overline{x}) \tag{4.2b}$$

where $g(x;\omega) := (\alpha_i(x),\ i \in \ell)$.

Proof. We observe that the impossibility of (4.2a) is equivalent to the optimality of $\overline{x}$, By Lemma 3.2, we have that $\overline{x}$ is optimal for (2.1) iff it is a solution of the selected problem (3.4). Taking into account (4.1), this is, in turn, equivalent to the impossibility of the system (4.2b). □

LEMMA 4.1 (Homogeneization) . *Let* ψ_0 *and* $-\psi_i, i \in \ell$ *be* $\mathcal{C}$*-differentiable with respect to the set of the* 2nd *and* 3rd *arguments. If* $\overline{x}$ *is a minimum point of* (1.2), *then there exist a non-negative SM* $\overline{\omega}(t) = (\overline{\omega}_i(t), i \in \ell) \in C^0(T)^m$ *and a neighbourhood in the sense of closedness of order one, say* $N^{(1)}(\overline{x})$, *such that the system* (*in the unknown* $\delta\overline{x} = x - \overline{x}; \delta\overline{x}' = x' - \overline{x}'$):

$$\int_T \mathscr{D}_{\mathcal{C}}\psi_0(t,\overline{x},x';\delta\overline{x},\delta\overline{x}')\mathrm{d}t < 0;$$

$$\int_T \overline{\omega}_i(t)\cdot\mathscr{D}_{-\mathcal{C}}\psi_i(t,\overline{x},\overline{x}';\delta\overline{x},\delta\overline{x}')\mathrm{d}t > 0, \quad i \in \ell^0, \tag{4.3}$$

$$g_i(\overline{x};\overline{\omega}_i) + \int_T \overline{\omega}_i(t)\cdot\mathscr{D}_{-\mathcal{C}}\psi_i(t,\overline{x},\overline{x}';\delta\overline{x},\delta\overline{x}')\mathrm{d}t \geqslant 0, i \in \ell\backslash\ell^0; x \in X \cap N^{(1)}(\overline{x}),$$

is impossible, where

$$\ell^0 := \left\{ i \in \ell : g_i(\overline{x};\overline{\omega_i}) = 0, \int_T \overline{\omega}_i(t)\cdot\epsilon_i(t,\overline{x},\overline{x}';\delta\overline{x},\delta\overline{x}')\mathrm{d}\not\equiv 0 \right\}.$$

Proof. By applying Proposition 4.1 we get the existence of $\overline{\omega}$ such that (4.2) is impossible. Now, ab absurdo, suppose that, at the same $\omega=\overline{\omega}$, (4.3) be possible, and let $\hat{x}\neq\overline{x}$ be a solution. Then $\alpha\hat{x}$ is a solution of (4.3) $\forall\alpha\in]0,1]$, since $g_i(\overline{x};\overline{\omega}_i)\geqslant 0$ and $\mathcal{D}_C f, \mathcal{D}_{-C}\psi_i, i\in\ell$ are positively homogeneous (satisfy (12a) of [3]). The assumption implies that the remainders:

$$\int_T \epsilon_{\psi_0}\mathrm{d}t, \int_T \overline{\omega}_i\epsilon_i \mathrm{d}t, \quad i\in\ell$$

are infinitesimal of order >1 with respect to $||(\delta\overline{x},\delta\overline{x}')||$, so that, setting $\hat{y}=(\hat{x},\hat{x}')$, $\overline{y}=(\overline{x},\overline{x}')$ and $\delta\overline{y}=(\hat{x}-\overline{x},\hat{x}'-\overline{x}')=(\delta\overline{x},\delta\overline{x}'), \exists\hat{\alpha}\in]0,1]$ such that:

$$\frac{1}{||\hat{\alpha}\delta\overline{y}||}\int_T \epsilon_{\psi_0}(t,\overline{y};\hat{\alpha}\delta\overline{y})\mathrm{d}t < -\frac{1}{||\delta\overline{y}||}\int_T \mathcal{D}_C f(t,\overline{y};\delta\overline{y})\mathrm{d}t,$$

$$\frac{1}{||\hat{\alpha}\delta\overline{y}||}\int_T \overline{\omega}_i(t)\cdot\epsilon_i(t,\overline{y};\hat{\alpha}\delta\overline{y})\mathrm{d}t > -\frac{1}{||\delta\overline{y}||}\int_T \overline{\omega}_i(t)\cdot\mathcal{D}_{-C}\psi_i(t,\overline{y};\delta\overline{y})\mathrm{d}t, \quad i\in\ell^0.$$

From these inequalities, by noting that $g_i(\overline{x};\overline{\omega}_i)=0, \forall i\in\ell^0$, we have:

$$\int_T [\mathcal{D}_C\psi_0(t,\overline{y};\hat{\alpha}\delta\overline{y})+\epsilon_f(t,\overline{y};\hat{\alpha}\delta\overline{y})]\mathrm{d}t<0, \tag{4.4a}$$

$$g_i(\overline{x};\overline{\omega}_i)+\int_T\{\overline{\omega}_i(t)[\mathcal{D}_{-C}\psi_i(t,\overline{y};\hat{\alpha}\delta\overline{y})+\epsilon_i(t,\overline{y}_i;\hat{\alpha}\delta\overline{y})]\}\mathrm{d}t>0, \quad i\in\ell^0. \tag{4.4b}$$

$\forall i\in\ell\backslash\ell^0$ either $g_i(\overline{x};\overline{\omega}_i)=0$ and $\int_T\overline{\omega}_i\epsilon_i\mathrm{d}t\equiv 0$ or $g_i(\overline{x};\overline{\omega}_i)>0$. In the former case, with $\hat{\alpha}=1$, we obviously have:

$$g_i(\overline{x};\overline{\omega}_i)+\int_T\{\overline{\omega}_i(t)[\mathcal{D}_{-C}\psi_i(t,\overline{y};\hat{\alpha}\delta\overline{y})+\epsilon_i(t,\overline{y};\hat{\alpha}\delta\overline{y})]\}\mathrm{d}t\geqslant 0. \tag{4.4c}$$

In the latter case $\exists\alpha^0\in]0,1]$ such that:

$$g_i(\overline{x};\overline{\omega}_i)+\int_T\overline{\omega}_i(t)\cdot\mathcal{D}_{-C}\psi_i(t,\overline{y};\alpha\delta\overline{y})\mathrm{d}t>0, \forall\alpha\in]0,\alpha^0],$$

and thus $\exists\tilde{\alpha}\in]0,\alpha^0]$ such that:

$$\frac{1}{||\tilde{\alpha}\delta\overline{y}||}\int_T\overline{\omega}_i(t)\cdot\epsilon_i(t,\overline{y};\tilde{\alpha}\delta\overline{y})\mathrm{d}t\geqslant$$

$$\geqslant -\frac{1}{\alpha^0||\delta\overline{y}||}\left[g_i(\overline{x};\overline{\omega}_i)+\alpha^0\int_T\overline{\omega}_i(t)\cdot\mathcal{D}_{-C}\psi_i(t,\overline{y};\delta\overline{y})\mathrm{d}t\right]\geqslant$$

$$\geqslant -\frac{1}{\alpha||\delta\overline{y}||}\left[g_i(\overline{x};\overline{\omega}_i)+\alpha\int_T\overline{\omega}_i(t)\cdot\mathcal{D}_{-C}\psi_i(t,\overline{y};\delta\overline{y})\mathrm{d}t\right], \quad \forall\alpha\in]0,\alpha^0],$$

where the 1st inequality holds since $\int_T \overline{\omega}_i \epsilon_i \mathrm{d}t$ is infinitesimal of order >1 with respect to $||(\delta\overline{x}, \delta\overline{x}')||$ and the 2nd side is fixed and negative, the 2nd inequality holds since the 2nd side is obviously the maximum of the 3rd on $]0, \alpha^0]$. With $\overline{\alpha} := \tilde{\alpha}$ it follows that:

$$g_i(\overline{x}; \overline{\omega}_i) + \int_T \{\overline{\omega}_i(t)[\mathscr{D}_{-C}\psi_i(t, \overline{y}; \hat{\alpha}\delta\overline{y}) + \epsilon_i(t, \overline{y}; \hat{\alpha}\delta\overline{y})]\}\mathrm{d}t \geqslant 0. \tag{4.4d}$$

Collecting all (4.4), recalling that $g_i(\overline{x}; \overline{\omega}_i) = 0, i \in \ell^0$, and using the definition of the remainders ϵ_i, we obtain the possibility of system (4.2), and hence the contradiction. This completes the proof. □

The impossibility of system (4.3) can be expressed as disjunction of the two sets of the image space associated to (1.2). To this end, introduce the sets:

$$\mathscr{H}_h := \{(u, v) \in \mathrm{IR} \times \mathrm{IR}^m : u > 0; v_i > 0, \quad i \in \ell^0; v_i \geqslant 0, \quad i \in \ell \backslash \ell^o\};$$

$$\mathscr{K}(\omega) := \{(u, v) \in \mathrm{IR} \times \mathrm{IR}^m : u = -\int_T \mathscr{D}_C f\, \mathrm{d}t; v_i = g_i(\overline{x}; \omega_i) +$$

$$+ \int_T \omega_i \mathscr{D}_{-C}\psi_i \mathrm{d}t, i \in \ell; x \in X\}.$$

It is easily seen that the impossibility of system (4.3) holds iff

$$\mathscr{H}_h \cap \mathscr{K}(\overline{\omega}) = \varnothing.$$

Note that system (4.3) is set up with the homogeneous parts of f and the selections g_i and hence $\mathscr{K}_h(\omega)$ represents the homogenization of the selected image $\mathscr{K}(\omega); \mathscr{H}_h$ simply follows the changes in the types of inequalities in going from (4.2a) to (4.3).

When $\psi_0, \psi_i, i \in \ell$ are differentiable ($\mathcal{C}$ is replaced with its subset $\mathcal{L}$ of linear elements), then (4.3) becomes:

$$\int_T [\langle \nabla_x \psi_0, x - \overline{x}\rangle + \langle \nabla_{x'} \psi_0, x' - \overline{x}'\rangle]\mathrm{d}t < 0;$$

$$\int_T \overline{\omega}_i [\langle \nabla_x \psi_i, x - \overline{x}\rangle + \langle \nabla_{x'} \psi_i, x' - \overline{x}'\rangle]\mathrm{d}t > 0, \quad i \in \ell^0;$$

$$g_i(\overline{x}; \omega_i) + \int_T \overline{\omega}_i [\langle \nabla_x \psi_i, x - \overline{x}\rangle +$$

$$+ \langle \nabla_{x'} \psi_i, x' - \overline{x}'\rangle]\mathrm{d}t \geqslant 0, i \in \ell \backslash \ell^0; x \in X \cap N^{(1)}(\overline{x}),$$

and in this case Lemma 4.1 extends to problem (1.2) a well known Linearization Lemma (see [1]). Note that Lemma 4.1 can be slightly sharpened by requiring differentiability or $\mathcal{C}$-differentiability only for those ψ_i such that $g_i(\overline{x};\overline{\omega}_i)=0$ and continuity for the remaining ones. Lemma 4.1 can be generalized to semidifferentiable functions.

5. Semistationarity

The generalization of the concept of stationary point, which is associated with that of necessary conditions, has received much attention. The crucial point is the kind of convergence that is required. The following definition seems to be quite general, even if it is clear that it is not possible to handle every problem with a single kind of convergence.

DEFINITION 1. $\overline{x}\in R\subseteq Y$ *will be called a lower semistationary point of a problem of type* $\min_{x\in R} f(x)$, *iff there exists a neighbourhood* $N(\overline{x})$ *of* $\overline{x}$, *such that:*

$$\liminf_{x\to\overline{x}} \frac{f(x)-f(\overline{x})}{||x-\overline{x}||}\geqslant 0, x\in N(\overline{x}). \qquad (5.1)$$

The following proposition, whose proof is in Sect. 3 of [3], is a motivation for adopting the above definition.

PROPOSITION 5.1.

(i) *If* $\overline{x}$ *is a minimum point of* f *on* R, *then (5.1) holds.*

(ii) *If* R *and* f *are convex, then a lower semistationary point of* f *on* R *is a global minimum point, and* (5.1) *becomes*:

$$f'(\overline{x};x-\overline{x})\geqslant 0, \quad \forall x\in R,$$

where $f'(\overline{x};z)$ *denotes directional derivative at* $\overline{x}$ *in the direction* z.

(iii) *If* f *is differentiable, then* (4.1) *becomes*:

$$\langle f'(\overline{x}),x-\overline{x}\rangle\geqslant 0, \quad \forall x\in R,$$

which if $\overline{x}\in intR$, *collapses to*:

$$f'(\overline{x})=0.$$

Note that, in the case of problem (1.2), f' denotes the variation of the functional f. Let us introduce the function:

$$L(x;\theta,\lambda,\omega):=\theta f(x)-\langle\lambda g(x;\omega)\rangle, \quad (\theta,\lambda)\in\mathrm{IR}\times\mathrm{IR}^m, \quad \omega\in\Omega.$$

Note that, if we set $\lambda_i(t) := \lambda_i \cdot \omega_i(t), L$ is the classic Lagrangian function associated to (1.2); hence, here the Lagrangian multiplier is splitted into two parts: a selection part, i.e., $\omega_i(t)$ which in a wider context becomes $\omega_i(t,x)$, and a separation part, i.e., λ_i.

We observe that the existence of the selection multipliers ω_i does not guarantee, in general, the existence of the classic Lagrange multipliers $\lambda_i(t)$, $i \in \ell$, as shown by the following example.

Example 5.1 Let us identify (1.2) with:

$$\min \int_T x(t)\mathrm{d}t, x^2(t) \leqslant 0, \forall t \in T := [0,1], x \in C(T).$$

Of course, $x(t) \equiv 0$ is the unique feasible (and hence optimal) solution. The selected problem (3.4), is

$$\min \int_T x(t)\mathrm{d}t, \int_T \omega(t) x^2(t)\mathrm{d}t \leqslant 0, x \in C(T),$$

where $\omega(t) \in C(T)$. If we choose $\omega(t) > 0$, $\forall t \in T$, then it is necessarily $x(t) \equiv 0$ to have $x(t)$ admissible for the selected problem; this, therefore, turns out to be equivalent to the given problem. It is simple to check that the selected problem does not admit the classic finite-dimensional Lagrange multipliers λ_i, associated to the separation part in the factorization of $\lambda_i(t)$, $i \in \ell$.

A star as apex of a cone will denote its positive polar. Let $\overline{k}(\omega) := (0, g(\overline{x}; \omega)) := (\overline{u}, \overline{v}(\omega))$ a selection of the image of $\overline{x}$. Unlike before, $y := (x, x'), \overline{y} := (\overline{x}, \overline{x})$, $\delta\overline{y} := y - \overline{y}$.

LEMMMA 5.1 (Semistationarity). *Let ψ_0 be $\mathcal{C}$-differentiable and $\psi_i, i \in \ell$ be (-$\mathcal{C}$)-differentiable with respect to the set of* 2nd *and* 3rd *arguments at any value of them.*

(i) *If* $\exists \overline{\omega} \in \Omega$ *such that*:

$$-(\theta, \lambda) \in [\mathscr{K}(\overline{\omega}) - \overline{k}(\overline{\omega})]^*, \tag{5.2}$$

then

$$\liminf_{x \to \overline{x}} \frac{L(x; \theta, \lambda, \overline{\omega}) - L(\overline{x}; \theta, \lambda, \overline{\omega})}{||x - \overline{x}||} \geqslant 0. \tag{5.3}$$

If $\lim\limits_{||\delta\overline{y}|| \downarrow 0} \mathscr{D}_{\mathcal{C}} \psi_0(t, \overline{y}; \frac{\delta\overline{y}}{||\delta\overline{y}||})$ *and* $\lim\limits_{||\delta\overline{y}|| \downarrow 0} \mathscr{D}_{-\mathcal{C}} \psi_i(t, \overline{y}; \frac{\delta\overline{y}}{||\delta\overline{y}||}), i \in \ell$ *exist, then the lower limit of* (5.1) *collapses to the ordinary limit.*

(ii) *If* $\overline{x} \in intX$ *and* $\psi_0, \psi_i, i \in \ell$ *are differentiable, then* (i) *becomes*: *if*

$$-(\theta, \lambda) \in [\mathscr{K}(\overline{\omega}) - \overline{k}(\overline{\omega})]^{\perp},$$

then

$$L'_x(\overline{x}; \theta, \lambda, \overline{\omega}) = 0.$$

Proof. (5.2) is equivalent to:

$$-(\theta,\lambda)\in\Big\{(u^*,v^*)\in \mathrm{IR}\times\mathrm{IR}^m:$$
$$\langle (u^*,v^*),(u-\overline{u},v(\overline{\omega})-\overline{v}(\overline{\omega}))\rangle\geqslant 0,\quad \forall (u,v(\overline{\omega}))\in\mathscr{K}(\omega)\Big\},$$

or, by using Proposition 3.1 of [5],

$$\mathscr{D}_C L(x;\delta\overline{x};\theta,\lambda,\overline{\omega})-\mathscr{D}_C L(\overline{x};\delta\overline{x};\theta,\lambda,\overline{\omega})\geqslant 0,\quad \forall x\in X, \tag{5.4}$$

where

$$\mathscr{D}_C L=\int_T[\theta\mathscr{D}_C\psi_0-\sum_{i\in\ell}\lambda_i\mathscr{D}_{-C}\psi_i]\mathrm{d}t.$$

Divide both sides of (5.4) by $||\delta\overline{x}||$
and add to them:

$$\frac{1}{||\delta\overline{x}||}\overline{\epsilon}(\overline{x};\delta\overline{x};\theta,\lambda,\overline{\omega}):=\frac{1}{||\delta\overline{x}||}\int_T\left(\theta\epsilon_{\psi_0}-\sum_{i\in\ell}\lambda_i\epsilon_i\right)\mathrm{d}t;$$

then (4.4) becomes:

$$\frac{1}{||\delta\overline{x}||}[L(x;\theta,\lambda,\overline{\omega})-L(\overline{x};\theta,\lambda,\overline{\omega})]\geqslant\frac{1}{||\delta\overline{x}||}\overline{\epsilon}(\overline{x};\delta\overline{x};\theta,\lambda,\overline{\omega}),\quad \forall x\in X\backslash\{\overline{x}\}.$$

Now (5.3) follows, since $\overline{\epsilon}/||\delta\overline{x}||\to 0$ as $x\to\overline{x}$. The remaining part is obvious.

(ii) Since $\mathscr{K}(\overline{\omega})$ is now affine, the polar becomes the orthogonal complement and hence liminf collapses to lim and this is zero since both $\geqslant$ and $\leqslant$ must hold.□

6. Concluding Remarks

We have considered a selection approach to extremum problems having an infinite-dimensional image.

The main feature of the proposed approach is to postpone the introduction of infinite-dimensional arguments, to the definition of the image of the problem in a finite-dimensional space. The infinite-dimensional nature of the problem arises in the analysis of the existence of a well-behaved selection of the image multifunction. In particular, it has been shown that the set of selection multipliers, that in general depend on the variable $x\in X$ (the given space), turns out to be an enlargement of the class of the Lagrange multipliers associated to the problem.

An open question is to establish conditions that guarantee the existence of selection multipliers independent on x (as supposed in Sect. 4), in order to recover, by means of the selection theory, the classic results of the Calculus of Variations.

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